Construction of the Natural Numbers from a Real Exponential Field

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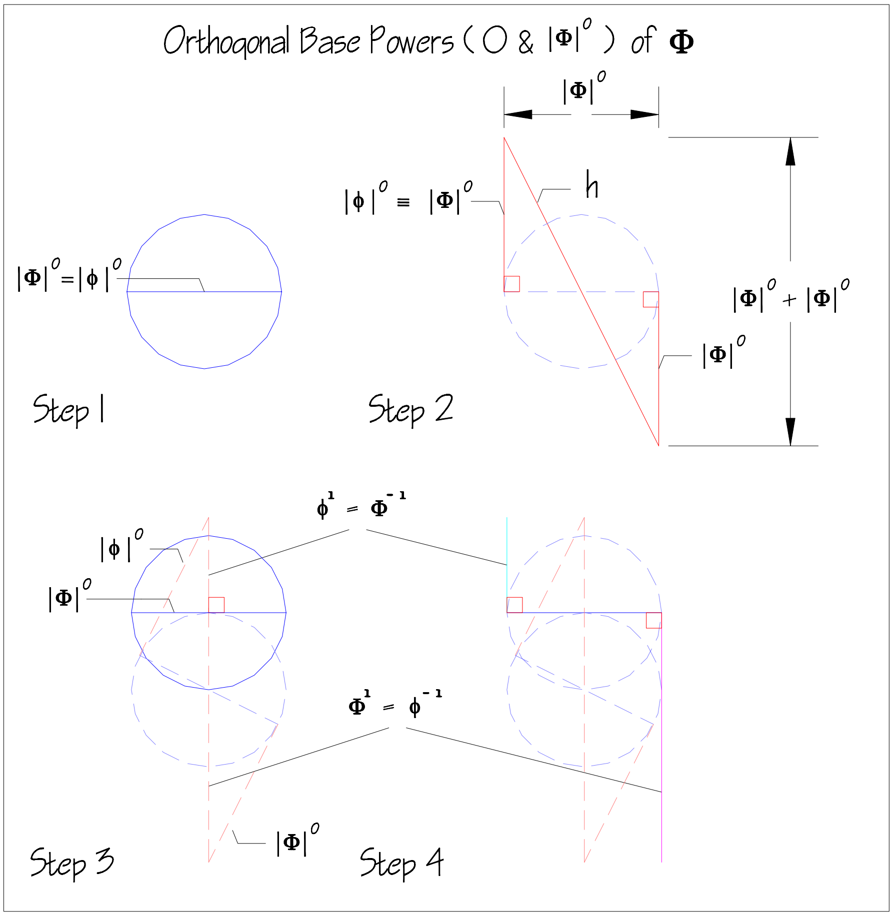
Consider the following geometric construction.



Step 1 – Construct a unit circle, that is a circle with unit diameter, *d*, and unit π circumference, *c*. Designate the unit base number, Φ, and the exponential unit, Φ^0. Designate the inverse unit base number, ϕ, and the inverse exponential unit, ϕ^0. Thus the magnitude of the diameter, which cannot be negative (and where the absolute brackets are added here for emphasis but will later be left off unless required for clarity), is

$$|d| = |\Phi|^0 = |\phi|^0 \tag{1.1}$$

It is noted that as exponential bases, these base numbers cannot be negative, though their exponents can be. We might also imagine that their exponential functions, $y_\Phi = f(x) = \Phi^x$ or $y_\phi = f(x) = \phi^x$, can be applied to a directional vector of any sense.

Step 2 – Extend two tangent line segments of unit length, orthogonal to and in opposite directions from opposing ends of the diameter. Construct a line segment joining the distal ends of the tangents, the hypotenuse, *h*, of a right triangle of sides $\Phi^0 + \Phi^0$ and Φ^0. The square of the hypotenuse is

$$h^2 = \left(\Phi^0 + \Phi^0\right)^{\Phi^0+\Phi^0} + \left(\Phi^0\right)^{\Phi^0+\Phi^0} \tag{1.2}$$

We will call $\Phi^0 = \phi^0$, the number 1 and $\Phi^0 + \Phi^0 = \phi^0 + \phi^0$, the number 2. Note, however, that

$$1 = \Phi^0 = \phi^0 \text{ does not necessarily equal } \left(\Phi^0\right)^{\Phi^0+\Phi^0} = \left(\phi^0\right)^{\phi^0+\phi^0} = 1^2 \tag{1.3}$$

in qualitative terms, since the first terms are linear intervals or spans of the real number line, while the second terms could represent a unit area, a span of the real number plane. Thus $\Phi^0 + \Phi^0 = 2$, as an exponent can denote an orthogonal condition, or a "square" number. It conveys a geometric component to the use of exponents via the Pythagorean theorem.

With respect to the extent of the unit circle above, then

$$c = \pi\left(\Phi^0\right) = \pi_c \tag{1.4}$$

while the extent of the sphere, *s*, with *c* as its extremal cross section is

$$s = \pi\left(\Phi^0\right)^2 = \pi_s \tag{1.5}$$

and it is obvious that, while the magnitudes of *c* and *s* are equal, qualitatively

$$\pi_c \neq \pi_s\,. \tag{1.6}$$

The above construction is equivalent to a rotation of Φ^0 about each end of the diameter, each through an angle of $+\frac{\pi}{2}$. Label the tangents by their magnitudes, the left tangent, $|\phi|^0$, and the right tangent, $|\Phi|^0$. Thus the hypotenuse can be thought of as a unit diameter stretched between and connecting the rotated ends of the tangents. Consider the following identity, in which the sense of the exponents indicates that these two terms are anti-parallel. This follows conventional use of imaginary exponents in which the sense of the exponent is rotational, where generally positive designates counterclockwise and negative, clockwise.

$$\phi^0 \equiv \Phi^{-0} \tag{1.7}$$

Here the imaginary notation in the exponent is masked, since a rotation of $+\frac{\pi}{2}$ is represented

$$\Phi^{+i} \tag{1.8}$$

and since the powers of exponents are additive, $\left(\Phi^{+i}\right)^2 = \Phi^{+i+i} = \Phi^{-}$

Instead as of rotation of diameters, we can also effect this construction with a rotation of two opposing, centrally directed radii, *r*, by moving their central points in opposite directions, orthogonally along the path of the resulting hypotenuse. We then have the condition of the dashed lines found in Step 3, which is the configuration of Step 2 rotated in the plane of the paper clockwise 0.4636476…radians. We find the radii have doubled their magnitudes in transforming to resulting tangents. The angle tangents of the distal vertices are correspondingly 1/2.

Step 3 – Centered at the intersection of the unit circle and the hypotenuse, adjacent to the tangent $|\phi|^0$, construct another unit circle with its diameter tangent to the first circle. The distance along the hypotenuse from this center to the vertex with $|\phi|^0$ will be

$$\phi^{\phi^0} \equiv \phi^1 = \Phi^{-\Phi^0} \equiv \Phi^{-1} \tag{1.9}$$

and the distance from that center along the hypotenuse to the other vertex, with the tangent $|\Phi|^0$ will be

$$\Phi^{\Phi^0} \equiv \Phi^1 = \phi^{-\phi^0} \equiv \phi^{-1}. \tag{1.10}$$

We now have the following identity

$$h^2 = \left(\Phi^0 + \Phi^0\right)^2 + \left(\Phi^0\right)^2 \equiv \left(\Phi^1 + \phi^1\right)^2 \tag{1.11}$$

Step 4 – Parallel to the hypotenuse and at the ends of the diameter of the second circle construct the tangents ϕ^1 and Φ^1. It is apparent from the above description that we have the following:

$$h = \sqrt{2^2 + 1^2} = \sqrt{5} \tag{1.12}$$

$$\phi^1 = \Phi^{-1} = \frac{\sqrt{5}}{2} - \frac{1}{2} = 0.618033989... \tag{1.13}$$

$$\Phi^1 = \phi^{-1} = \frac{\sqrt{5}}{2} + \frac{1}{2} = 1.618033989... \tag{1.14}$$

The above steps have created the base for an exponential, orthogonal expansion from a linear unitary condition. From a most fundamental rational operation of division, dividing a whole, 1, into halves, or of multiplication, doubling of a unit or 2, involving the most fundamental ratio, 1/2, we have arrived at what some have termed the most irrational of numbers. It is significant, however, that the primary triad of this particular rational group does not form a group under ordered addition, i.e.

$$a + b = c \tag{1.15}$$

where $a < b < c$, since

$$\tfrac{1}{2} + 1 \neq 2 \tag{1.16}$$

From another perspective, however, it is ½ and 2 which are the result of the rational operation of the bases Φ and ϕ, where

$$\phi = \frac{1}{\Phi} \tag{1.17}$$

This primary rational triad <u>does</u> form a group under ordered addition, since

$$\phi + 1 = \Phi \tag{1.18}$$

which can be generalized using the base Φ to the ordered orthogonal addition

$$\Phi^{-1} + \Phi^0 = \Phi^1 \tag{1.19}$$

and further, where q is any real number, to

$$\Phi^{q-1} + \Phi^q = \Phi^{q+1} \tag{1.20}$$

In fact (1.17) comprises the only set of three numbers, where

$$a = \frac{b}{c} \tag{1.21}$$

that satisfies the condition of (1.15). It is also of interest to our discussion that (1.17) as (1.21) is of the general form

$$\frac{1}{x^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{x}. \tag{1.22}$$

This implies that in some fashion, ϕ and Φ, respectively serve as surrogates for $\sqrt{1}$ and 1^2.

From (1.7) we can see that changing the rotational sense of the base results in mirror symmetry, while changing the ccw and cw conventions results in an inversion or rotation of π of the whole system.

This would be mildly interesting in itself, but when we add a third geometric dimension and generalize the linear powers of the base Φ, we make some compelling discoveries. By linear powers is meant a mapping of the exponential function of any base to the real number line. By contrast, a geometric mapping of *n*-integer exponents maps to an *n*-dimensional space, as indicated above.

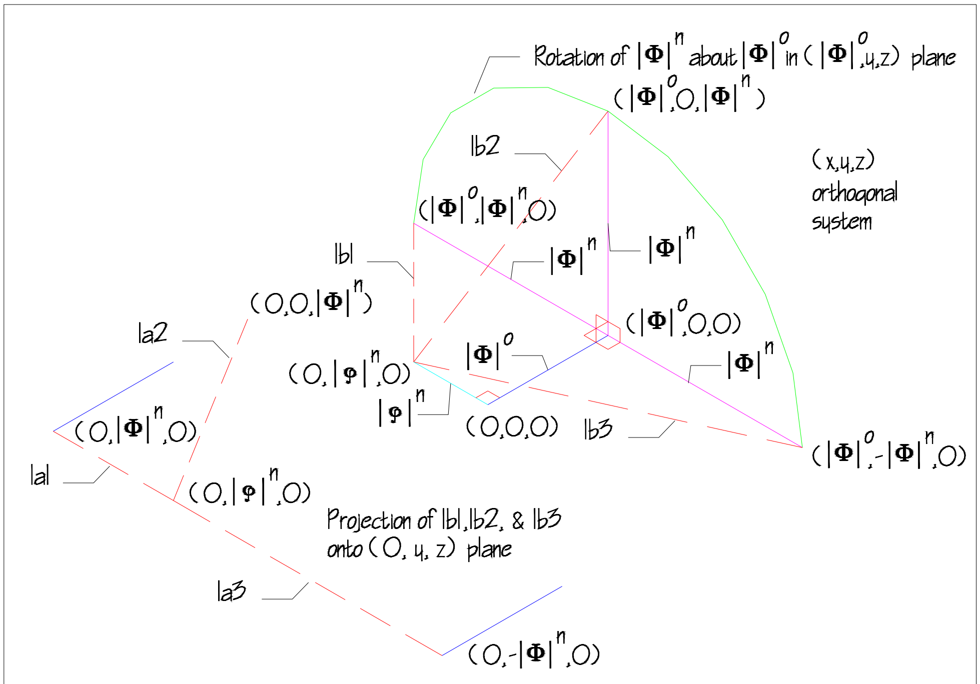
In the above development, *h* is presented as a stretching or augmentation of the length of a unit diameter, Φ^0, for a unit π circle or 1-sphere manifold. It is also the diameter of a unit π 2-sphere manifold embedded in a 3-D Euclidean space, as mentioned. We could have drawn the steps shown on a unit square or 1-cube with an implied unit 2-cube in a 3-D space, in which case Step 2 would depict a 2-cube collapsed across one set of diagonals, as a box flattened under foot.

We can therefore think of $\Phi^{-1}\left(=\phi^{1}\right), \Phi^{0}\left(=\phi^{-0}\right),$ and $\Phi^{1}\left(=\phi^{-1}\right)$ as three bases of an orthogonal, exponential space. The unit 3-D space thus spanned has the same volume as a unit cube or 1^3. The magnitude of the vector sum of the three bases, however, instead of $\sqrt{3}$ as in the unit cube, in this case is $\sqrt{4} = 2$. For the purposes of this development, we will stipulate that any augmentation of the bases is exponential and not by addition or multiplication, that the middle base will remain a unit base, and that the volume spanned, that is the product of the three components, remains 1^3. That is

$$\Phi^{-q}\Phi^{0}\Phi^{q} = 1 \tag{1.23}$$

For that case in which $q = 0$, this remains the standard unit cube.

We will further limit ourselves to the condition where $q = n =$ an integer. It is immediately clear that a negative integer simply reverses the bases ϕ and Φ. We are interested in three orthogonal conditions, with two readings of each, as shown in the following diagram. Condition *b* consists of *b1*, in which ϕ and Φ are parallel, *b2*, in which ϕ and Φ are orthogonal, and *b3*, the condition established by Step 4 above, in which ϕ and Φ are anti-parallel. Condition *a* consists of the projection of *b1*, *b2* and *b3* into the plane normal to Φ^0, giving us *a1*, *a2* and *a3* respectively.



Orthogonal n Powers of $\mathbf{\Phi}$

We are interested in the solutions for these 6 conditions for the values of *n*. The table below gives the first few of these, where the 6 conditions are identified as follows:

$$a1 \equiv \Phi^n - \phi^n \tag{1.24}$$

$$b1 \equiv \left[\left(\Phi^n - \phi^n\right)^2 + \left(\Phi^0\right)^2\right]^{\frac{1}{2}} \tag{1.25}$$

$$a2 \equiv \left[\left(\Phi^n\right)^2 + \left(\phi^n\right)^2\right]^{\frac{1}{2}} \tag{1.26}$$

$$b2 \equiv \left[\left(\Phi^n\right)^2 + \left(\Phi^0\right)^2 + \left(\phi^n\right)^2\right]^{\frac{1}{2}} \tag{1.27}$$

$$a3 \equiv \Phi^n + \phi^n \tag{1.28}$$

$$b3 \equiv \left[\left(\Phi^n + \phi^n\right)^2 + \left(\Phi^0\right)^2\right]^{\frac{1}{2}} \tag{1.29}$$

We have the following sequence, where the integers shown are the square of *a1*, *b1*, etc…for Φ to the n, thus should be read "the square root of (the integer) is *a1*, *b1*, etc…", for the first 13 integral values of *n*.

n		*a1*	*b1*	*a2*	*b2*	*a3*	*b3*
0		0	1	**2**	3	4	5
1		1	2	**3**	4	5	6
2		5	6	**7**	8	9	10

3	16	17	**18**	19	20	21
4	45	46	**47**	48	49	50
5	120	122	**123**	124	125	126
6	320	321	**322**	323	324	325
7	841	842	**843**	844	845	846
8	2205	2206	**2207**	2208	2209	2210
9	5776	5777	**5778**	5779	5780	5781
10	15125	15126	**15127**	15128	15129	15130
11	39601	39602	**39603**	39604	39605	39606
12	103680	103681	**103682**	103683	103684	103685

The bold figured values of *a2* are (the square roots of) the even numbered elements of the Lucas series, starting at 0. This gives the linear values or lengths of the hypotenuse developed above through the various orthogonal exponential transformations outlined above. The square of these values therefore maps these integers to the real number <u>plane</u>.

From this relationship we might create a numerical system on the Φ base,

$$ {}_a\Phi_b^n(n) \tag{1.30}$$

where *n* is the "decimal" place and the condition *a* or *b* might be indicated by a number or a sense sign, such as + = parallel, i = orthogonal, and - = anti-parallel, so that for $n = 0$

$$ {}_+\Phi^0 = {}_1\Phi^0 = 0 \tag{1.31}$$

$$ \Phi_+^0 = \Phi_1^0 = 1 \tag{1.32}$$

$$ {}_i\Phi^0 = {}_2\Phi^0 = 2 \tag{1.33}$$

$$ \Phi_i^0 = \Phi_2^0 = 3 \tag{1.34}$$

$$ {}_-\Phi^0 = {}_3\Phi^0 = 4 \tag{1.35}$$

$$ \Phi_-^0 = \Phi_3^0 = 5 \tag{1.36}$$

Thus the number 1000 is variously represented, among other possibilities, as

$$ \begin{gathered} {}_i\Phi^7\ {}_i\Phi^5\ {}_i\Phi^3\ {}_+\Phi^3 = \Phi_-^7\ \Phi_-^5\ \Phi_-^3\ \Phi_-^1\ \Phi_-^0 \\ 843+123+18+16 = 846+126+21+6+1 \end{gathered} \tag{1.37}$$

The significance of this and the reason for its development here is conceptual and related to number theory. While the real number line is often represented as the "naturals" then by elaboration filled in with the "rationals", then finally made continuous by filling in the gaps, here we start by using a decidedly "irrational" base to generate a real plane continuum, by squaring the length of the hypotenuse of our continually transforming radial legs, from which the integers and thereby the rationals emerge as a function of the orthogonal positioning of the legs. This shows yet another geometric and orthogonal interpretation of exponentiation.