Excerpt of Appendix D – Exponentiation and a System of Natural Logarithms

from

A Classical Complex 4-Wave Foundation

of the

Cosmic-Quantum Mechanism

-

Fundamental Rest Mass Quanta as
Simple Harmonic Oscillations of
the Spacetime Continuum
at Resonant Frequency and Wave Number,
Driven by Cosmic Expansion

By Mart Gibson

October 14, 2006
Revisions and Addition of the Section on The Quantum Metric
as of February 1, 2007

Mart Gibson
P.O. Box 2358
Southern Pines, NC 28388
910-692-7462
martin@uniservent.org

Copyright © Martin Gibson 2021 All Rights Reserved

Appendix D - Exponentiation

Calculus is the study of the rate of change in one variable quantity, conventionally denoted by a *y*, which is held to be a function, *f*, wholly or partially, of another variable, generally denoted by an *x* or sometimes a *t*. This underlying functional relationship between the variables is denoted by

$$y = f(x) \text{ or } y = f(t). \tag{7.23}$$

In the case of a partial function, a function of more than one variable, we write

$$y = f(x,t). \tag{7.24}$$

Thus, with (7.23) when $x = a$, $y = b$, and with (7.24) when $x = a$ and $t = b$, $y = c$. *a*, *b* and *c* are arbitrary symbols standing for unknown quantities of the stated variable *x*, *t* and *y*, and depending on the context and circumstance *a*, *b* and *c* may in fact be the same or of equal value.

The underlying functional relationship or function does not necessarily indicate that *x* causes *y* or that *y* is the operational function of *x*. While this may be so in the case of some physical and organizational conditions, in general terms the function simply indicates that when *x* has the value of *a*, *y* is always, within the context determined by *f*, uniquely observed to have the value of *b*.

Thus, given a right triangle of variable angle, α, but fixed, unit length hypotenuse, the cosine can be stated as a function of the length of the adjacent side, *a*, and the length of *a* can be stated as an inverse function of the cosine. In the language of mathematics, we would say that the cosine function maps the value of *a* onto the cosine and the inverse function maps the value of the cosine onto *a*. This concept of mapping reflects the fact that any function that we might consider can be visualized and charted against the backdrop of an orthogonal co-ordinate system.

Thus it may equally be true that

$$x = f(y) \text{ or } t = f(y). \tag{7.25}$$

Note that it is not generally stated that

$$x,t = f(y), \tag{7.26}$$

although that may in fact be the case. If *x* is the adjacent side of α and *t* is the opposite, they both vary with respect to some variation in the angle, $y = \alpha$.

While it may be of interest to know the value of *y* for any value of *x* or *t*, it is often of equal or greater interest to know the rate at which *y* is changing for any value of *x* or *t*. This rate of change or ratio of variability of one quantity with respect to another is known as the derivative function, f', of *y* with respect to *x* or *t*, or

$$\frac{dy}{dx} = f'(x) \text{ or } \frac{dy}{dt} = f'(t). \tag{7.27}$$

The quantity *dy* is the differential amount of change in *y* that occurs for every differential amount of change, *dx,* in *x*. While *dy* and *dx* are customarily envisioned as being infinitesimally small, they are generally not small by the same proportions, and are

indeed expressed as the change in *y* in units of that quality for every change of one unit of *x*. Hence they are often used in partial derivative form as direction cosines, where by implication, ∂x would be the hypotenuse of a right triangle of unit value and ∂y is the adjacent side. Here f' constitutes another function of *x*, with the prime notation indicating that it is a derivative function of *f(x)*. The single prime is the rate of change in *f(x)* commensurate with a change in *x*. If it is a function with respect to time, *f(t),* i.e. a rate of change over some unit of time, it is a speed, or if a direction is specified, a velocity.

If the rate of change in *f(x)* or *f(t)* is not steady or constant, then we have a second derivative of these functions, denoted by a double prime

$$\frac{d^2y}{dx^2} = f''(x) \text{ or } \frac{d^2y}{dt^2} = f''(t). \tag{7.28}$$

The change in velocity, acceleration, which is the second derivative with respect to time, $f''(t)$, is commonly encountered and understood. The equivalent with respect to *x,* also a type of acceleration, is a change in the intensity or magnitude of some derivative, $f'(x)$, with each change in *x*. Thus if $f'(x)$ represents the slope of a mountainside, the change in elevation per change in horizontal displacement, when the slope is a constant pitch, then $f''(x) = 0$, that is, it does not exist. If the slope gets steeper as the mountain is climbed, then the second derivative is positive. In physics this second derivative of *x* is called the Laplacian. A force embodying the inverse square law is an instance of the second derivative.

If the acceleration, the change in $f'(x)$ or $f'(t)$, is not constant, then we have a third derivative of these functions, denoted by the triple prime or

$$\frac{d^3y}{dx^3} = f'''(x) \text{ or } \frac{d^3y}{dt^3} = f'''(t). \tag{7.29}$$

With respect to $f''(t)$, this acceleration of acceleration is known as jerk. Any change from a position of rest involves an element of jerk, since the acceleration from zero to any finite velocity is not instantaneous or constant. With respect to a mountainside, if the slope increases exponentially with the climb, instead of at a steady rate of say 50 meters per kilometer of horizontal distance covered, then the third derivative is functioning.

In (7.28) and (7.29) it will be noticed that the differential with respect to *y* is preceded by the order of the derivative, while the differential with respect to *x* and *t* is followed by the order or exponent of the derivative. This is due to the fact that the latter variables are actual squares and cubes, that is powers of the differentials, while the order of the dependent differential of *y* indicates the change in *y* attributed to the independent variable of the same order. The dimensionality of *y* is always of what ever happens to be the inherent dimensionality of the quality *y* represents. If *y* is a force, $d^n y$ will itself have units of force. Notice that $f^n(t)$, then would have units of force per time to the n^{th} power.

A geometric example will perhaps make this clear. While the following may not be the customary context for the second and higher order derivatives, it is a legitimate instance of such. The equation for the volume of a cube, in which the volume, V, is a function of the length of one side, all sides being by definition equal, is

$$V = f(x) = x^3. \tag{7.30}$$

The inverse function is

$$x = f(V) = V^{\frac{1}{3}}. \tag{7.31}$$

A change in volume, dV, is still a three dimensional quality, and we might be tempted to say that it is equal to dx^3, which in a certain context it is. If we want to express that change as a derivative function, however, we must use the definition of a derivative, which gives the original function plus the differential change as

$$V + dV = (x + dx)^3 = x^3 + 3x^2dx + 3xdx^2 + dx^3. \tag{7.32}$$

Subtracting (7.30), the terms in square brackets, gives the differential

$$V + dV - [V] = x^3 + 3x^2dx + 3xdx^2 + dx^3 - [x^3] \tag{7.33}$$

$$dV = 3x^2dx + 3xdx^2 + dx^3 \tag{7.34}$$

This last equation is a bit opaque, however, as it assumes that one of the corners of the cube is at the origin of a co-ordinate system and injects a corresponding bias into the derivation, which may or may not be warranted. If we locate the origin at the center of cube, while still assuming each side aligned with the co-ordinates, we must assign the differential change in x to each end of a length and we have instead of (7.32)

$$\begin{aligned} V + dV - [V] &= (x + 2dx)^3 - [x^3] \\ &= x^3 + 6x^2dx + 12xdx^2 + 8dx^3 - [x^3] \end{aligned} \tag{7.35}$$

$$dV = 6x^2dx + 12xdx^2 + 8dx^3 \tag{7.36}$$

From this last derivation, it is immediately clear that the differential volume is made up of the six sides of the cube, the twelve edges, and the eight vertices or corners, all times an infinitesimal length of varying orders. The first order derivative, corresponding to the six square sides of the cube of measure x^2 and the one usually deemed to have the greatest significance, is explicitly stated as

$$f'(x) = \frac{d^1V}{dx^1} = 6x^2. \tag{7.37}$$

The second order derivative, corresponding with the twelve edges of the cube, consisting of twelve line segments of length x, is

$$f''(x) = \frac{d^2V}{dx^2} = 12x^1. \tag{7.38}$$

The third order derivative, corresponding to the eight vertices, each of zero, or technically, vanishing dimension, is

$$f'''(x) = \frac{d^3V}{dx^3} = 8x^0. \tag{7.39}$$

Thus for the total derivative or (increasing) change in V with respect to a change in x, we have

$$\sum_{n=1}^{3}\frac{d^nV}{dx^n}=\frac{d^1V}{dx^1}+\frac{d^2V}{dx^2}+\frac{d^3V}{dx^3}=6x^2+12x+8. \tag{7.40}$$

If V were decreasing, and dx were a decrement, the change would be

$$-\sum_{n=1}^{3}\frac{d^nV}{dx^n}=-\frac{d^1V}{dx^1}+\frac{d^2V}{dx^2}-\frac{d^3V}{dx^3}=-6x^2+12x-8. \tag{7.41}$$

Note that the decrease from the initial condition is indicated by the negative sense of the summation, but that the magnitude of the sum in this case is of a positive 6 squares, minus 12 line segments, while adding back the 8 vertices. In more elucidating fashion the magnitude of the derivative becomes

$$\sum_{n=1}^{3}\frac{d^nV}{dx^n}=\frac{d^1V}{dx^1}-\left(\frac{d^2V}{dx^2}-\frac{d^3V}{dx^3}\right)=6x^2-\left(12x-8\right). \tag{7.42}$$

While it is clear that the contribution of the 8 vertices is not effected by the value of x, it is obvious that as x increases, the contribution to the sum made by the edges increases linearly with x, while the contribution made by the surface squares increases exponentially, specifically by the power of 2.

If dx is not quite zero, but exceedingly small compared to x, then it is apparent that the order of the derivatives is a fair appraisal of each component's contribution to dV. The additional volume is predominantly surface differential. In fact, at the limit as dx approaches 0, each order is exponentially greater that the next order in succession. However, if the components are allowed to increase exponentially beyond the value of x, the situation inverts itself.

If we think of the original cube, still positioned about the origin, as having some very small unit edges of length x, where x is the smallest imaginable length, and make the change, dx, exceedingly great, in fact approaching infinity, then the first order of six squares constitutes the six sense-axes of a 3-D space, the second order, the twelve edges, constitutes the twelve quadrants of the x-y, y-z, and z-x planes, while the third order of the eight vertices become the 3-D octants of the co-ordinate system.

In the above cubic scenario, it is apparent that the numerical coefficients, which in a standard development of the calculus arise through the operation of the binomial expansion as with (7.32), are actually inherent aspects of the specific cubic geometry. The derivation consists of a division of each of the orders of differentiation by x, n times, where n indicates the order of each term. As such it represents a reduction in the power of each term by 1 for each time or order. Alternatively, we can view this as a multiplication for each instance of differentiation of $x^{-1}dx$. Thus, with the observation that a change in the volume of a cube must occur at its 3 boundary elements, i.e. faces, edges and vertices, (7.36) can be arrived at by

$$dV = \sum_{n=1}^{3} \frac{d^n V}{dx^n} dx^n = \sum_{n=1}^{3} \frac{{}^n V}{x^n} dx^n = \sum_{n=1}^{3} {}^n V \left(\frac{dx}{x} \right)^n$$
$$= 6x^3 \left(\frac{1}{x} dx \right)^1 + 12x^3 \left(\frac{1}{x} dx \right)^2 + 8x^3 \left(\frac{1}{x} dx \right)^3 \tag{7.43}$$
$$= 6x^2 dx + 12x dx^2 + 8dx^3$$

Notice that this describes an arrangement of 27 cubes, 3 x 3 x 3, with the original cube of volume V at the center, and that the total number of elements in the surface is 26, corresponding to the 26 adjacent cubes. The process of differentiation reduces each of these cubes exponentially, according to its relationship to the center cube. The n in ${}^n V$ indicates both the magnitude and the geometry of the coefficients that arise through the polynomial expansion. As there are two boundaries to any interval x, and an interval of equal magnitude to x at each boundary, x_b, a binomial of power n is

$$\left(x + 2x_b \right)^n = 1x^n + \left(3^n - 1 \right) x^{n-m} x_b^m. \tag{7.44}$$

As an example, a 4-D hypercube expansion is

$$\left(x + 2x_b \right)^4 = 1x^4 + 80x^{n-m} x_b^m = 1x^4 + 8x^3 x_b^1 + 24x^2 x_b^2 + 32x^1 x_b^3 + 16x^0 x_b^4. \tag{7.45}$$

Making the substitution, $\lim_{x_b \to 0} x_b = dx$ for the case of a 4-D equivalent to (7.43) gives

$$\left(x + 2dx \right)^4 - x^4 = 80x^{n-m} dx^m = 80x^n \left(\frac{dx}{x} \right)^m$$
$$= 8x^4 \left(\frac{dx}{x} \right)^1 + 24x^4 \left(\frac{dx}{x} \right)^2 + 32x^4 \left(\frac{dx}{x} \right)^3 + 16x^4 \left(\frac{dx}{x} \right)^4 \tag{7.46}$$
$$= 8x^3 dx^1 + 24x^2 dx^2 + 32x^1 dx^3 + 16x^0 dx^4$$

As this is obviously a logarithmic operation, as indicated by the bracketed terms, where

$$x^n = y \;\; \therefore \;\; \mathbf{log}_x \, y = n \tag{7.47}$$

and as the derivative of the natural log is

$$d \, \mathbf{ln} \, x = d \, \mathbf{log}_e \, x = \frac{1}{x} dx, \tag{7.48}$$

we can recast (7.43) as

$$dV = \sum_{n=1}^{3} \frac{d^n V}{dx^n} dx^n = \sum_{n=1}^{3} {}^n V \left(d \, \mathbf{ln} \, x \right)^n$$
$$= 6x^3 \left(d \, \mathbf{ln} \, x \right)^1 + 12x^3 \left(d \, \mathbf{ln} \, x \right)^2 + 8x^3 \left(d \, \mathbf{ln} \, x \right)^3 \tag{7.49}$$

This last observation suggests a fundamental tie-in between the derivative of a polynomial function and that of the natural logarithm.

With this in mind, it is apparent that the matter of differentiation is closely related to the subject of exponentiation, and appears to consist of reduction by one power or order of exponentiation for each order of differentiation. In the case of the cube, it is clear that

each order indicates an orthogonal reduction, from cube to square, to line segment to point.

We might wonder if this is the case for all functions of the form of (7.23). We might, for example, have a function that relates the perimeter of a square to its edge. The resulting equation and its derivative are

$$\begin{aligned} P &= 4x \\ P + dP - \left[P\right] &= 4(x + dx) - \left[4x\right] \\ dP &= 4dx \\ \frac{dP}{dx} &= 4 \end{aligned} \tag{7.50}$$

What is different in this case, is that there is no apparent second derivative. In other words, for

$$\frac{dP}{dx} = f'(x) = 4 \tag{7.51}$$

the function $f'(x)$ is not affected by the value of x. It is a constant and does not involve a rate of change of $f'(x)$ with respect to a change in x.

Any function that does not change has a first order derivative of 0. This does not mean, however, that it might not have a second order derivative. Metaphorically speaking, the water swirling around a drain might be described by some function that maps its motion around the plane of the water's surface. At the point at which it becomes vertical and disappears down the drain, effectively leaving the dimensional space of the tub, the derivative with respect to change in that space vanishes. Obviously there still must be some function describing its motion vertically and perhaps even horizontally once it has entered the plumbing system, albeit, in terms of that other dimension. The key is to realize that (7.51) actually should be written

$$\frac{dP}{dx} = f'(x) = 4x^0 = 4(1) = 4\,\mathbf{ln}_x\, x \tag{7.52}$$

$$\begin{aligned} \therefore \frac{d^2P}{dx\,d\,\mathbf{ln}_x\, x} &= 4 \\ \frac{d^2P}{dx^2} &= f''(x) = \frac{4}{x} \end{aligned} \tag{7.53}$$

There is another condition, however, in which, although there is a changing rate of change, there is no <u>apparent</u> change in the rate of change, i.e. no <u>apparent</u> second derivative, and that is the case of the exponential function, specifically of the natural base e, inversely related to the natural logarithmic function. The exponential function is its own derivative, of whatever order we might envision, where

$$f(x) = y = e^x \text{ iff } \mathbf{ln}\, y = x \tag{7.54}$$

so that

$$f'(x) = D_x\left[f(x)\right] = D_x e^x = e^x \tag{7.55}$$

where D_x is a differential operator, that is, it operates on f to produce f'. Thus since

$$D_x'' e^x = D_x \left[D_x e^x \right] = D_x \left[e^x \right] = e^x \tag{7.56}$$

the difference between any two orders of differentiation, in fact, between any order and the exponential function itself is

$$D_x'' e^x - D_x e^x = D_x e^x - e^x = e^x - e^x = 0. \tag{7.57}$$

and

$$\frac{e^x}{D_x^n e^x} = 1 \tag{7.58}$$

We might wonder what significance this has, since subtracting one order derivative from another is rather like subtracting oranges from apples. They are two different types of entity, just as a point is a different type of entity than a line segment or length, which is itself different from an area, itself different from a volume. In fact, there are generally held to be an infinite number of points in a length, lengths in an area, and areas in a volume, i.e. of dimensions in the next higher order of dimension, so the subtraction of the lesser from the greater leaves the latter substantially unchanged.

This then is the point in (7.58). The dimensional identity of each order of differentiation is the same as that of the basic function, $f(x) = y$, since it is the exponent itself that is variable and not the base. If we compare this condition with that of the cube and its derivative orders, in which the relative contribution of each order to the overall change in V is dependent on the ratio of dx^n:dx, we see that for an exponential change, the relative contribution of each order at the limit is unaffected by the change in x, or as it is often the variable used in this context, t. The ratio in the exponential case is always 1, hence the apparent lack of change.

A doubling of the sum of the lengths of the edges and a quadrupling of each surface area results in an eightfold increase in the volume of the cube. Note that there is no change in the number of boundary elements and their angular configuration, which is the defining condition of the cube. Using a combinatorial or additive approach to creating a change in the cube, then, we see that it is more economical to augment the edges to move the vertices further apart, which defines the volume change, than to fill the cube with volume, since a unit of length is orders of magnitude less than a unit of volume. Yet there is no conformal or topological difference between a cube of unit volume and one of volume 8, something we intuitively understand. The matter of scale only attains significance within a combinatorial or economical approach.

Hence in a continuum analysis, where the elements of various dimensions are integrally related, i.e. non-combinatorially, if we had a cube experiencing a continuous exponential change, each of the elements in its boundary, the faces, edges and vertices, would increase proportionally to its order with the change in volume, each derivative order increasing in proportion as the exponent of x or t. In such event, using the value of $x = \sqrt[3]{V}$ as our standard, all orders show the same exponential change and the whole is relatively, or perhaps better stated, intrinsically unchanged. It is only within the context

of some extrinsically determined property, such as some external standard of length or density, i.e. volume, surface or linear, that change is registered or observed.

This is intuitively understood, especially in the preparation of scaled engineering drawings and models and other graphic representations. It is also known rarely to occur in the physical world in which physical forms result from the combination of discrete units or building blocks of matter. Adult humans do not generally look like babies three to four times their original length. Equally true, most tree girth-to-height ratios increase with growth. On the other hand, most celestial bodies of any size assume a generally spherical shape, irrespective of their volume. Rephrasing (7.43)

$$\begin{aligned} dV = \sum_{n=1}^{3} \frac{d^n V}{dV^{\frac{n}{3}}} dV^{\frac{n}{3}} = \sum_{n=1}^{3} \frac{{}^n V}{V^{\frac{n}{3}}} dV^{\frac{n}{3}} = \sum_{n=1}^{3} {}^n V \left(\frac{dV^{\frac{1}{3}}}{V^{\frac{1}{3}}} \right)^n \\ = 6V \left(\frac{dV^{\frac{1}{3}}}{V^{\frac{1}{3}}} \right)^1 + 12V \left(\frac{dV^{\frac{1}{3}}}{V^{\frac{1}{3}}} \right)^2 + 8V \left(\frac{dV^{\frac{1}{3}}}{V^{\frac{1}{3}}} \right)^3 \\ = 6\left(V^{\frac{3}{3}-\frac{1}{3}} \right) dV^{\frac{1}{3}} + 12\left(V^{\frac{3}{3}-\frac{2}{3}} \right) dV^{\frac{2}{3}} + 8\left(V^{\frac{3}{3}-\frac{3}{3}} \right) dV^{\frac{3}{3}} \\ = 6V^{\frac{2}{3}} dV^{\frac{1}{3}} + 12V^{\frac{1}{3}} dV^{\frac{2}{3}} + 8V^{\frac{0}{3}} dV^{\frac{3}{3}} \end{aligned} \tag{7.59}$$

From another but still exponential perspective, in terms of our initially outlined derivative orders, this indicates that the magnitude of displacement, velocity, acceleration, and jerk might be equal or

$$f'''(x) - f''(x) = f''(x) - f'(x) = f'(x) - f(x) = 0. \tag{7.60}$$

This is essentially the same equation as (7.57) and indicates that a relationship of this type is exponential in nature.

We do not have to look far for another familiar instance of such. While change is generally equated with motion and thereby with displacement or translational change of position, rotation presents an instance of motion without translational displacement, taking the position of the rotating body as a whole. It is in a sense change without change, and is elegantly presented using the Euler identity, which involves the exponential expansion of an imaginary logarithm or

$$e^{ix} = \mathbf{cos}\, x + i\, \mathbf{sin}\, x = y \tag{7.61}$$

As with the exponential function of (7.54), the domain of x is the real number line, but in this case, instead of the range of $y > 0$, y oscillates over the range of $-1 \le y \le 1$ if we consider only the real component, for each change in x of 2π, our angles and the "natural" unit of x in this case being in radians. Otherwise y must be a complex number whose

range is a circle centered on the *x*-*y* origin in the complex plane, in this case of implied radius $r = 1$. If we then map *y* to the real *x*-*y* plane, by multiplying *y* times its complex conjugate, where *y* then gives the radius *r* and *x* is the count of the rotations, the range of *y* will be a horizontal line crossing the *y* axis at $y = r = 1$. Note that a circle of fixed radius from the origin maps as a horizontal line segment of 2π length. Since the derivative of such a line is zero, the rate of change of the rotation is zero, at least in this mapping. The only variable in such case might be the velocity of the rotation which might be reflected in the scale, the density, of the real number line. Presumably a denser placement of the integers would represent a greater velocity.

The use of a radian as the unit measure of *x* makes the equation self normalizing, that is, it sets the *x* and *y* axes of a co-ordinate system against which we might plot the function to the same scale. Assuming a rotational amplitude or modulus, i.e. the radius, *r*, equal to the hypotenuse, selection of a unit value for the *y* axis for a cosine of 1 automatically dictates the unit length for the *x* axis, since a radius, *r*, and a rotational arc of one radian measured at a distance *r* from the center are of equal length. We can then envision *x* as the distance traveled by a point *P* on the circumference of a rotating disk or equator of a sphere of radius *r*, but it might simply be a point in space that is revolving about some center of oscillation which is also the polar origin.

Thus for any value of *x*,

$$\frac{x}{2\pi} = n + \frac{\varphi}{2\pi} \tag{7.62}$$

where *n* is an integer number of rotational cycles, $-\infty < n < +\infty$, and where φ is a remainder angle or phase in which $0 < \varphi < 2\pi$. As we shall soon see, we might also state

$$\frac{x}{\frac{\pi}{2}} = n + \frac{2\varphi}{\pi} \tag{7.63}$$

where *n* is the count of the number of times that $|y|$ equals one.

If we want to express *x* in some conventional unit such as meters, we simply multiply it by the number of meters per radian and *x* will be in units of meters. In such case, in order to convert *x* back to normalized units, (7.61) becomes

$$e^{i\kappa x} = \mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta = y \tag{7.64}$$

where κ is the angular wave number or number of arc radians per unit of length and

$$\kappa = \frac{\theta}{x}. \tag{7.65}$$

A similar approach for *t* gives us

$$e^{it} = \mathbf{cos}\,t + i\,\mathbf{sin}\,t = y \tag{7.66}$$

where time is in natural units or radians, and for conversion from conventional units of time,

$$e^{i\omega t} = \mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta = y \tag{7.67}$$

where ω is the angular frequency or number of arc radians per unit of time and

$$\omega = \frac{\theta}{t}. \tag{7.68}$$

Such a condition would apply to a standing wave of fixed angular frequency.

For a wave of fixed frequency traveling from a propagating source, we can combine the two to get

$$e^{i\theta} = e^{i(\kappa x + \omega t)} = \mathbf{cos}\,\theta + i\,\mathbf{sin}\,\theta = y \tag{7.69}$$

where it is understood that x and t are of ambivalent sense.

Finally by extending the exponent of e to complex numbers we have,

$$e^{r\pm i\theta} = e^{r}e^{\pm i\theta} = e^{r}e^{\pm i(\kappa x+\omega t)} = R\left(\mathbf{cos}\,\theta \pm i\,\mathbf{sin}\,\theta\right) = y \tag{7.70}$$

$$e^{-r\pm i\theta} = e^{-r}e^{\pm i\theta} = e^{-r}e^{\pm i(\kappa x+\omega t)} = \frac{1}{R}\left(\mathbf{cos}\,\theta \pm i\,\mathbf{sin}\,\theta\right) = y, \tag{7.71}$$

and we can see that (7.69) is simply a special case of these last two in which $R = 1$. Assuming that $R > 1$, using $y = r = \sqrt{a^2 - (ib)^2}$, (7.70) now maps to the x-y real plane as a horizontal line greater than $y = 1$, and (7.71) maps to a line between the x axis and the line $y = 1$. The argument, as the middle term of (7.64) is called, conceals the fact that the sense of the angle θ determines whether i$\mathbf{sin}\,\theta$ is positive, changing in a counterclockwise sense, or negative, in a clockwise sense. Thus we could apply a convention in which the negative sense of i$\mathbf{sin}\,\theta$ maps (7.70) and (7.71) to horizontal lines crossing the negative y axis. The rotation velocity and frequency would then switch sense.

Let us examine the function

$$y = f(W) = We^{W}. \tag{7.72}$$

Inverting the function so that

$$W = f(y) \tag{7.73}$$

gives

$$f(y)e^{f(y)} = y. \tag{7.74}$$

We can find that

$$f(y)e^{f(y)} = W(n)e^{W(n)} = y \tag{7.75}$$

where $W(n)$ is related to the Lambert W function and in fact is identical to that function for the principal branch, integer values $n > 0$, or Lambert W(0, n>0). As will be shown, while the Lambert W function is complex for all values of $n < 0$, for all values of $-\infty \le n \le +\infty$ $W(n)$ is real and $W(n)$ = - $W(-n)$. Further, we define

$$W(n) = n\,\mathbf{ln}\,x. \tag{7.76}$$

Therefore, (7.75) becomes

$$n\,\mathbf{ln}\,xe^{n\,\mathbf{ln}\,x} = n\,\mathbf{ln}\,x\left(x^{n}\right) = U_{n}n = y. \tag{7.77}$$

where U_n is the co-efficient of n needed to produce x for any value of y.

Inverting the function to its original

$$y = nx^n \ln x \tag{7.78}$$

and differentiating gives

$$\begin{aligned} dy &= \left(nx^n\right) d\ln x + \left(n\ln x\right) nx^{n-1} dx \\ &= \left(nx^n\right)\frac{dx}{x} + \left(n\ln x\right) nx^{n-1} dx \end{aligned} \tag{7.79}$$

with the derivative

$$\frac{dy}{dx} = \left(1 + n\ln x\right) nx^{n-1} \tag{7.80}$$

If we substitute for the natural log derivative, we have instead

$$\frac{dy}{d\ln x} = \left(1 + n\ln x\right) nx^n . \tag{7.81}$$

From (7.76) it follows that

$$\frac{dy}{dx} = \left(1 + W\left(n\right)\right) nx^{n-1} \tag{7.82}$$

and with a little foresight, we might imagine that this hides a complex function, as with

$$\begin{aligned} \frac{dy}{dx} &= \left(1 + iW\left(n\right)\right) nx^{n-1} \\ \frac{dy}{d\ln x} &= \left(1 + iW\left(n\right)\right) nx^n \end{aligned} \tag{7.83}$$

Returning to (7.76) for that case in which $U_n = 1$, $y = n$ and the normalized value of $W(n)$ is

$$W_0\left(n\right) = n\ln x_n , \tag{7.84}$$

where the n in the subscript of x relates that value of x as the unique normalizing value for $W_0\left(n\right)$ and we have

$$W_0\left(n\right)\left(x_n^n\right) = n\ln x_n\left(x_n^n\right) = n = \frac{y}{U_n}. \tag{7.85}$$

It follows that

$$\frac{W_0\left(n\right)}{n} = \ln x_n = \frac{1}{x_n{}^n} = \kappa_n{}^n = \frac{y}{nU_n x_n{}^n} . \tag{7.86}$$

As before, substituting t for x, the equivalent for (7.86) is

$$\frac{W_0\left(n\right)}{n} = \ln t_n = \frac{1}{t_n{}^n} = \omega_n{}^n = \frac{y}{nU_n t_n{}^n} \tag{7.87}$$

Finally, continuing in that vein

$$\omega_n = \frac{1}{t_n} = \left(\frac{W_0(n)}{n} \right)^{\frac{1}{n}} = \frac{1}{x_n} = \kappa_n \qquad (7.88)$$
$$= \left(\mathbf{ln}\, t_n \right)^{\frac{1}{n}} = \left(\mathbf{ln}\, x_n \right)^{\frac{1}{n}}$$

Since for that case in which $n = 0$, (7.86) becomes

$$\mathbf{ln}\, x_0 = \frac{W_0(0)}{0} = \frac{1}{x_0{}^0} = \kappa_0{}^0 = 1, \qquad (7.89)$$

apparently

$$x_0 = e = e_0 \text{ and } \kappa_0 = e^{-1} = e_0^{-1} \qquad (7.90)$$

and the dividend in the first term of (7.86) must be a continuous function which approaches 0 as n approaches 0. Thus for any n, we have a fundamental base e_n in which it can be stated

$$\mathbf{ln}\, x_n = \mathbf{ln}_0\, e_n = \frac{W_0(n)}{n} = e_n{}^{-n} = e_{-n}{}^{n} = e_{\kappa n}{}^{n}. \qquad (7.91)$$

and for the inverse of x_n

$$\mathbf{ln}\, x_n^{-1} = \mathbf{ln}_0\, e_n^{-1} = \mathbf{ln}_0\, e_{-n} = \frac{W_0(-n)}{n} = -e_n{}^{-n} = -e_{-n}{}^{n} = -e_{\kappa n}{}^{n}. \qquad (7.92)$$

where we define, for conceptual reasons,

$$e_n{}^{-n} \equiv e_{-n}{}^{n} \equiv e_{\kappa n}{}^{n}. \qquad (7.93)$$

It is noted that the natural log in the second term is specified to apply to the case of (7.89) so that for any value x, for the conventional natural log x, or $\mathbf{ln}\, x$,

$$\mathbf{ln}\, x = \mathbf{ln}_{\substack{n \\ n=0}}\, x = \mathbf{ln}_0\, x. \qquad (7.94)$$

Thus for any e_n, we have

$$\mathbf{ln}_n\, e_n = e_n{}^{n}\, \mathbf{ln}_0\, e_n = 1 \qquad (7.95)$$

and

$$\mathbf{ln}_n\, x = e_n{}^{n}\, \mathbf{ln}_0\, x = y = U_n n. \qquad (7.96)$$

In continuation, we have

$$e_n^{y} = e_0^{\left(\frac{y}{e_n^{n}} \right)} = e_0^{\,y e_{-n}{}^{n}} = x, \qquad (7.97)$$

$$e_0{}^{y} = e_n^{\,y e_n{}^{n}} = e_{-n}^{\,-y e_n{}^{n}} = e_n^{\,y e_{-n}{}^{-n}} = e_{-n}^{\,-y e_{-n}{}^{-n}}, \qquad (7.98)$$

$$e_0{}^{-y} = e_{-n}^{\,y e_n{}^{n}} = e_{-n}^{\,y e_{-n}{}^{-n}} = e_n^{\,-y e_n{}^{n}} = e_n^{\,-y e_{-n}{}^{-n}} \qquad (7.99)$$

and

$$\mathbf{ln}_0\, x = \frac{\mathbf{ln}_n\, x}{e_n{}^{n}} = e_{-n}{}^{n}\, \mathbf{ln}_n\, x = e_{-n}{}^{n} y = e_{-n}{}^{n} U_n n, \qquad (7.100)$$

It follows with respect to derivatives that

$$\frac{1}{x}=\frac{d\,\mathbf{ln}_0\,x}{dx}=e_{-n}{}^{n}\frac{d\,\mathbf{ln}_n\,x}{dx}. \tag{7.101}$$

It is further noted that

$$\mathbf{ln}_{-n}\,x=\left(\mathbf{ln}_n\,x\right)^{-1} \tag{7.102}$$

For all integer values of $n>0$, we redefine (7.85) and have

$$y=nU_n=n\,\mathbf{ln}_0\,x\left(x^n\right) \tag{7.103}$$

$$nx^n=\frac{nU_n}{\mathbf{ln}_0\,x}=\frac{y}{\mathbf{ln}_0\,x} \tag{7.104}$$

$$x^n=\frac{U_n}{\mathbf{ln}_0\,x}=\frac{y}{n\,\mathbf{ln}_0\,x} \tag{7.105}$$

and

$$\mathbf{ln}_0\,x=\frac{U_n}{x^n}=\frac{y}{nx^n} \tag{7.106}$$

Multiplying (7.104) by (7.101), gives

$$D_x\left(x^n\right)=nx^{n-1}=\left(\frac{1}{x}\right)nx^n=\frac{nU_n}{\mathbf{ln}_0\,x}\left(\frac{e_{-n}{}^{n}d\,\mathbf{ln}_n\,x}{dx}\right)=\frac{y}{\mathbf{ln}_0\,x}\left(\frac{e_{-n}{}^{n}d\,\mathbf{ln}_n\,x}{dx}\right) \tag{7.107}$$

and it can be seen that the terms on the right are equal to the derivative of x^n.

With some rearrangement we have

$$x^n\left(n\frac{dx}{x}\right)=\frac{U_n}{\mathbf{ln}_0\,x}\left(ne_{-n}{}^{n}d\,\mathbf{ln}_n\,x\right)=\frac{y}{n\,\mathbf{ln}_0\,x^n}\left(ne_{-n}{}^{n}d\,\mathbf{ln}_n\,x\right). \tag{7.108}$$

Since the terms in brackets in (7.108) are equal, it is apparent that the differential of any variable x of order n, of any function $f(x^n)$, is the product of that function and

$$n\frac{dx}{x}=nd\,\mathbf{ln}_0\,x=ne_{-n}{}^{n}d\,\mathbf{ln}_n\,x=W_0\left(n\right)d\,\mathbf{ln}_n\,x \tag{7.109}$$

where the term $ne_{-n}{}^{n}$ is equal to $W_0(n)$ and to the principal branch value of the Lambert W function for n. It follows that the derivative of x with respect to the nth natural log is

$$\frac{dx}{d\,\mathbf{ln}_n\,x}=\frac{xd\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x}=e_{-n}{}^{n}x=\frac{W_0\left(n\right)}{n}x \tag{7.110}$$

which normalized to the nth power would be

$$1_n=\frac{dx}{xd\,\mathbf{ln}_n\,x}=\frac{d\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x}=e_{-n}{}^{n}=\frac{W_0\left(n\right)}{n} \tag{7.111}$$

A factor for normalizing to the 0th power, therefore, would be, $e_n{}^n$, remembering (7.93), giving

$$1_0=e_n{}^{n}\frac{dx}{xd\,\mathbf{ln}_n\,x}=e_n{}^{n}\frac{d\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x}=e_n{}^{n}e_{-n}{}^{n}=e_n{}^{n}\frac{W_0\left(n\right)}{n}=\frac{W_0\left(n\right)}{n}\left(e_0{}^{W_0(n)}\right). \tag{7.112}$$

where the last term is of the form of (7.75).

For a negative derivative

$$-1_n = \frac{-dx}{xd\,\mathbf{ln}_n\,x} = \frac{-d\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = -e_{-n}{}^{n} = \frac{W_0\left(-n\right)}{n} \tag{7.113}$$

the factor is - $e_n{}^n$, so that

$$\begin{aligned} 1_0 &= -e_n{}^n \frac{-dx}{xd\,\mathbf{ln}_n\,x} = -e_n{}^n \frac{-d\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = -e_n{}^n\left(-e_{-n}{}^n\right) \\ &= -e_n{}^n \frac{W_0\left(-n\right)}{n} = \frac{W_0\left(-n\right)}{n}\left(-e_0{}^{W_0(n)}\right) \end{aligned} \tag{7.114}$$

If we now introduce a rotational (imaginary) element into this condition, from (7.86) and the above development, we have

$$e_n{}^i = e_0{}^{ie_{-n}{}^n} = e_0{}^{\frac{W(in)}{n}} \;\therefore e_n{}^{in} = e_{in}{}^n = e_0{}^{ine_{-n}{}^n} = e_0{}^{W(in)} \tag{7.115}$$

$$e_n{}^{-i} = e_0{}^{-ie_{-n}{}^n} = e_0{}^{\frac{W(-in)}{n}} \;\therefore e_n{}^{-in} = e_{-in}{}^n = e_0{}^{-ine_{-n}{}^n} = e_0{}^{W(-in)} \tag{7.116}$$

in which (7.115) and (7.116) are complex conjugates, each representing a unit vector in the complex plane, so that

$$e_n{}^{in} e_n{}^{-in} = 1 + i0\,. \tag{7.117}$$

It follows that

$$i1_n = \frac{idx}{xd\,\mathbf{ln}_n\,x} = \frac{id\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = ie_{-n}{}^n = \frac{W_0\left(in\right)}{n}. \tag{7.118}$$

implying

$$iW\left(n\right) = W\left(in\right). \tag{7.119}$$

Thus with substitution from (7.96), using the normalizing factor -$ie_n{}^n$, (7.118) becomes

$$\begin{aligned} 1_0 &= -ie_n{}^n \frac{idx}{xd\,\mathbf{ln}_n\,x} = -ie_n{}^n \frac{id\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = \left[-ie_n{}^n ie_{-n}{}^n\right] \\ &= -ie_n{}^n \frac{W_0\left(in\right)}{n} = \frac{W_0\left(in\right)}{n}\left(-ie_0{}^{W_0(n)}\right) \end{aligned} \tag{7.120}$$

Similarly for a clockwise rotation,

$$-i1_n = \frac{-idx}{xd\,\mathbf{ln}_n\,x} = \frac{-id\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = -ie_{-n}{}^n = \frac{W_0\left(-in\right)}{n} \tag{7.121}$$

we have the normalizing factor, $ie_n{}^n$, and

$$\begin{aligned} 1_0 &= ie_n{}^n \frac{-idx}{xd\,\mathbf{ln}_n\,x} = ie_n{}^n \frac{-id\,\mathbf{ln}_0\,x}{d\,\mathbf{ln}_n\,x} = \left[ie_n{}^n\left(-ie_{-n}{}^n\right)\right] \\ &= ie_n{}^n \frac{W_0\left(-in\right)}{n} = \frac{W_0\left(-in\right)}{n}\left(ie_0{}^{W_0(n)}\right) \end{aligned} \tag{7.122}$$

In the above treatment of (7.118) through (7.122), we have used real normalizing factors with imaginary sense. It is further noted that the normalizations shown in the square brackets of (7.120) and (7.122), where the left $\mp e_n{}^n$ operates on the right initial condition, $\pm e_{-n}{}^n$, are instances of a type of complex inversion and are therefore conformal and are not instances of complex conjugation as shown in (7.117). In this latter case we interpret the subscript as the real exponent, *n*, of the *n*th exponential base and the superscript as the rotational or "imaginary" exponent. In this regards we will find that

$$e_n{}^{in} = e_{in}{}^n \tag{7.123}$$

so the determining indication for the rotational exponent is the presence of the *i* sense. Thus we have the following complex identities

$$e_n{}^{in} \equiv e_{-n}{}^{-in} \equiv e_{in}{}^n \equiv e_{-in}{}^{-n} = \overline{e_n{}^{-in}} \equiv \overline{e_{-n}{}^{in}} \equiv \overline{e_{in}{}^{-n}} \equiv \overline{e_{-in}{}^n} \tag{7.124}$$

where the terms on the right are the complex conjugates of those on the left, all of which represent unit vectors whose points lie on the unit circle. Thus it is not to be interpreted in the usual sense of a decomposed complex number, since

$$z_e = e^n + e^{in} \neq e_n{}^{in} = \left(e^n\right)^{in} \equiv \left(e^{in}\right)^n. \tag{7.125}$$

We might also surmise that the above normalizations are analytic. The normalizing factor inverts first with respect to sense of the *n*th degree of the exponential base, e_{-n}, on the unit circle, which amplifies the modulus or vector length of that base. This changes (7.117) to

$$e_n{}^{in} e_{-n}{}^{in} = e_0{}^{in} = 1 + i\,\mathbf{sin}\,\theta. \tag{7.126}$$

This is followed by inversion in the real axis to get

$$e_0{}^{-in} e_n{}^{in} e_{-n}{}^{in} = e_0{}^{-in} e_0{}^{in} = e_0{}^0 = 1 + i\,\mathbf{sin}\,\theta + \left(0 - i\,\mathbf{sin}\,\theta\right) = 1 + i0. \tag{7.127}$$

where it is clear that the 0th exponential base raised to the 0th power is a unit vector on the real, *x* axis. Thus complex normalization in this instance amounts to

$$e_{-n}{}^{in} e_n{}^{-in} = e_0{}^0 = 1 + i0 \tag{7.128}$$

and complex inversion to

$$\frac{1 + i0}{e_{-n}{}^{in}} = \frac{e_0{}^0}{e_{-n}{}^{in}} = e_n{}^{-in} \tag{7.129}$$

which is an amplitwist as defined by Tristan Needham in Visual Complex Analysis. This is the case of the bracketed term of (7.120), which might be represented by multiplication of a point in the counterclockwise interior of the unit circle by its reflection in that circle followed by multiplication of the resulting vector and its complex conjugate. In the case of (7.122), we have such multiplication of a clockwise interior vector by a counterclockwise exterior vector, both cases resulting in a unit vector along the real axis.

With respect to (7.89), we see that what appears at first glance to be a singularity is in fact an identity of the 0th order. Remembering that the natural log function maps to the *y* axis and is therefore equivalent to the imaginary axis in the complex plane and i**sin** θ, using the normalization factor, $e_0{}^0 = 1$, and recalling that $x_n = e_n$

$$e_0^{+i0}\,\mathbf{ln}\,x_0 = e_0^{-i0} = \frac{W_0(0)}{0}e^{W(0)} = x_0^{+i0}x_0^{-i0} = 1 \pm i0. \tag{7.130}$$

The sense of the 0^{th} powers can be seen as a vector potential or direction, similar to the assignment of charge sense in a static electrical or potential field.

Investigation will show that for any n or q, real or imaginary,

$$\left(\frac{dx}{xd\,\mathbf{ln}_0\,x}\right) = \frac{W(n)}{n}e^{W(n)} = \frac{W(q)}{q}e^{W(q)} \tag{7.131}$$

Applying the above normalization factors to (7.83), we have

$$\begin{aligned} -ie_0^{W(n)}\frac{dy}{dx} &= \left(-ie_0^{W(n)} + W(n)e_0^{W(n)}\right)nx^{n-1} = \left(-ie_n^{\,n} + n\right)nx^{n-1} \\ -ie_0^{W(n)}\frac{dy}{d\,\mathbf{ln}\,x} &= \left(-ie_0^{W(n)} + W(n)e_0^{W(n)}\right)nx^{n} = \left(-ie_n^{\,n} + n\right)nx^{n} \end{aligned}. \tag{7.132}$$

The interpretation of this development is that while the q^{th} exponential base to the qth power maps the real number line x to the positive real number line y, the q^{th} exponential base to the iqth power or the iq^{th} exponential base to the qth power maps the real number line to the unit circle. Further, whereas there is an asymptote for the former in the direction of the negative x axis, the unit circle is both asymptote and tangent in either sense for the rotational mapping of e.

With respect to the integer orders of e, it is apparent that each represents a mapping to the real number line of an exponential change in n orthogonal spaces. Thus referring to (7.88) in the context of (7.112), we can state the following normalizations of an n dimensional t or x

$$\begin{aligned} 1 = e_n\omega_n = \frac{e_n}{t_n} &= \left(\frac{W_0(n)}{n}\right)^{\frac{1}{n}}\left(e_n^{\,n}\right)^{\frac{1}{n}} \text{ and} \\ &= e_n\left(\mathbf{ln}\,t_n\right)^{\frac{1}{n}} \end{aligned} \tag{7.133}$$

$$\begin{aligned} 1 = e_n\kappa_n = \frac{e_n}{x_n} &= \left(\frac{W_0(n)}{n}\right)^{\frac{1}{n}}\left(e_n^{\,n}\right)^{\frac{1}{n}}. \\ &= e_n\left(\mathbf{ln}\,x_n\right)^{\frac{1}{n}} \end{aligned} \tag{7.134}$$

Fleshing this out for the first 4 orders of n with conjectured generalization at infinity, we have the following table, as generated by Maple, where it can be seen that a negative n is simply an inversion of $e_n^{\,n}$ to $e_n^{-n} = e_{-n}^{\,n}$.

$f(n)$ | n	0	1	2	3	…	∞
e_n	2.718281828..	1.763222834..	1.531584394..	1.419024454..		1
$e_{-n}=e_n^{-1}$	0.367879441..	0.567143291..	0.652918640..	0.704709490..		1
e_n^n	1	1.763222834..	2.345750756..	2.857390779..		∞
$e_{-n}^n = e_n^{-n}$	1	0.567143291..	0.426302751..	0.349969632..		0
$\ln_0 e_n$	1	0.567143291..	0.426302751..	0.349969632..		0
$\ln_0 e_{-n}$	-1	-0.567143291..	-0.426302751..	-0.349969632..		0
$\ln_n e_n$	1	1	1	1		1
$\ln_n e_{-n}$	-1	-1	-1	-1		-1
$\ln_0 e_n^n = W(n)$	0	0.567143291..	0.852605502..	1.049908893..		∞
$\ln_0 e_{-n}^n = W(-n)$	0	-0.567143291..	-0.852605502..	-1.049908893..		-∞
$\ln_n e_n^n$	0	1	2	3		∞
$\ln_n e_{-n}^n$	0	-1	-2	-3		-∞

n	$f(n)$	$n \ln_0 e_n$	$in \ln_0 e_n$	$n \ln_0 ie_n$	$in \ln_0 ie_n$
1		0.5671…	i0.5671…	0.5671…$+i\pi/2$ $(=+1\ i\pi/2)$	$-\pi/2$ + i0.5671…
2		0.8526…	i0.8526…	0.8526…$+i\ \pi$ $(=+2\ i\pi/2)$	$-\pi$ + i0.8526…
3		1.0499…	i1.0499…	1.0499…$+i3\pi/2$ $(=+3\ i\pi/2)$	$-3\pi/2$ + i1.0499…
4		1.2021…	i1.2021…	1.2021…$+i\ 2\pi$ $(=+4\ i\pi/2)$	-2π + i1.2021…
5		1.3067…	i1.3067…	1.3067…$+i5\pi/2$ $(=+5\ i\pi/2)$	$-5\pi/2$ + i1.3067…
6		1.4324…	i1.4324…	1.4324…$+i\ 3\pi$ $(=+6\ i\pi/2)$	-3π + i1.4324…

In this final table, it is clear that the integers, n, are the count of the rotations of ½ π and of the powers and hence the number of orders of i, both indications of a degree of orthogonal structure.

The special case of e_2 is shown to be of fundamental significance to an understanding of the foundations of quantum mechanics. Thus

$$\begin{aligned} e_2 &= \left(\mathbf{ln}\, e_2\right)^{-\frac{1}{2}} \\ &= \frac{1}{\omega_2} = \partial t = \left(\mathbf{ln}\, \partial t\right)^{-\frac{1}{2}} \\ &= \frac{1}{\kappa_2} = \partial x = \left(\mathbf{ln}\, \partial x\right)^{-\frac{1}{2}} \\ &= 1.531584394\mathbf{...} \end{aligned} \tag{7.135}$$

With respect to a conservative 3-field, here shown for a stress, where the volume potential energy density is conserved, a logarithmic change of the tension fields in one dimension, leading to a change of opposite sense in the two shear fields,

$$\left(\mathbf{ln}\,\partial T_{\xi}\right)\left(\partial T_{\eta}\right)\left(\partial T_{\zeta}\right) = E_1 = 1 \tag{7.136}$$

is therefore

$$\partial T_{\eta,\zeta} = \left(\mathbf{ln}\,\partial T_{\xi}\right)^{-\frac{1}{2}} \tag{7.137}$$

which can be stated by using the coefficients as

$$e_2 \partial T_0 = \left(\mathbf{ln}\, e_2\right)^{-\frac{1}{2}} \partial T_0 . \tag{7.138}$$

Assuming the change in the tension field is less than unity results in a negative logarithm and an orthogonal (imaginary) sense to the transverse fields, giving us

$$i e_2 \partial T_0 = \left(\mathbf{ln}\, e_2^{-1}\right)^{-\frac{1}{2}} \partial T_0 = i1.531584394\ldots \partial T_0 . \tag{7.139}$$

Related excerpt with chart from:

A Condensed Matter Model of Fundamental Particle Genesis
as a Function of an Accelerating Cosmic Spacetime Expansion

2e — The Missing Mass of Beta Decay

We are not quite through with our investigation. While the ratio of neutron-electron mass as developed here is compelling, there is still a matter of the missing mass of beta decay. According to the CODATA ratios, the difference between the neutron-electron mass ratio and proton-electron mass ratio is

$$m_n/m_e - m_p/m_e = 1838.6836... - 1836.1526... = 2.5310... \tag{1.69}$$

Since the relative mass of the electron in this case is 1, there is a relative mass or equivalent energy of 1.530… that is unaccounted for and sometimes referred to as the missing mass of beta decay. If it is assumed that mass is a property that is somehow bound up in the confines of a discrete particle, this is a puzzlement. However, if it is understood to be a measure of the resistance of stress to a straight-line force, therefore a measure of redirection of energy as wave action and of curvature of spacetime strain, the problem vanishes.

The entire waveform of a neutron and proton is confined within the nucleus of an atom in condensed matter and the particle wavelength, as represented by the reduced Compton wavelength, is bound by rotation of the oscillation as developed previously, so that the wave energy is bound to the transverse or anti symmetric component of the wave tensor, except for the relatively minor symmetric components of quantum gravity. The electron wave form has both rotational and translational components, particularly at the time of beta decay, so that the differential changes of exponential expansion stress which generate the decay and the beta particle or electron have both of these components. We can restate the last equation with a little more detail, where the inertial constant as developed previously is used to define mass,

$$\frac{\eth}{\lambda\!\!\!^{-}_{C,n}} - \frac{\eth}{\lambda\!\!\!^{-}_{C,p}} - \frac{\eth}{\lambda\!\!\!^{-}_{C,e_{ij}}} = 0 \tag{1.70}$$

The reduced Compton wavelength of the electron is written with subscripts, *ij*, to indicate the symmetric and anti symmetric components of the wave form.

Now consider the function

$$W(n) = \ln_0 e_n^n, \; where \;\; \ln_n e_n^n = n \tag{1.71}$$

which is related to the Lambert W function, where n can be any real number, though we will only be considering the integers. The significant feature of this function is that it generates a system of natural logs, $\ln_n$, and corresponding exponential bases, e_n, that can be used as normalizing factors, so that

$$\ln_n e_n = 1, \;\; \ln_n e_{-n} = -1 \tag{1.72}$$

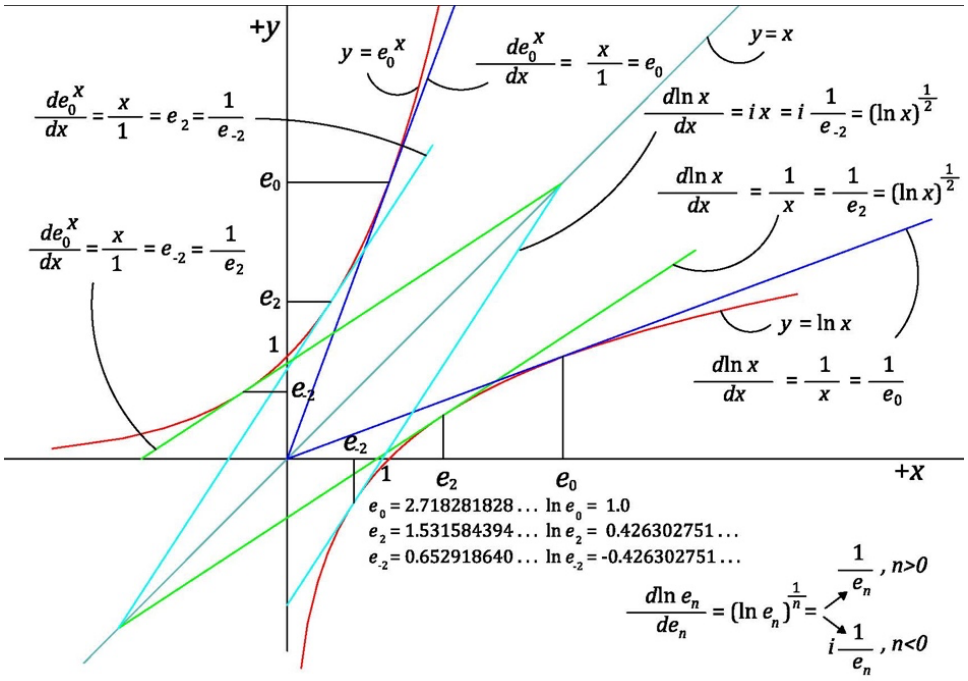
At $n(0)$, this is simply the natural log and exponential base, and

$$W(0) = \ln_0 e_0^0 = 0 \tag{1.73}$$

In the following Figure 11 we have graphed the significant portion of the natural log and exponential functions. Note the functions mirror each other along the line $y = x$, as do their derivatives. We can define the exponential base, e_0, on both x and y axes by the point on each function at which the lines (blue) whose slopes represent the derivatives intersect each other and the origin of the system. The only other instances of such intersection would be when the functions reach negative infinity along both axes, which of course they never do in the context of Euclidean space. They do on the Riemannian complex sphere, however.

The whole system of e_n, for $n > 0$, occurs in the range $1 < x \; and \; y < e_0$, and as n increases, the slope values converge while their intersection moves toward the negative infinities of both x and y. At the points on the curves corresponding to $x = 1$ for the logarithmic and $y = 1$ for the exponential, the derivatives of both

functions equal 1 and their slopes are parallel. (In terms of the Riemannian sphere, the slope lines for these derivatives (not shown here) actually form two ellipses about the spherical great circle $x = y$, tangent to a circle centered on the origin of radius r = 0.707… at (-0.5, +0.5) and (+0.5, -0.5) and approaching the +/-x and +/-y infinity intersection asymptotically.) As n decreases the slope intersections shift from negative to positive infinity at their x and y axis asymptotes. Thus $n < 0$ occurs in the range $0 < x$ *and* $y < 1$.



System of exponent bases e_n, shown for e_0 and $e_{\pm 2}$ where $e_{-2} = e_2^{-1}$

Figure 11

$f(n)$ n	0	1	2	3	…	∞
e_n	2.718281828..	1.763222834..	1.531584394..	1.419024454..		1
$e_{-n}=e_n^{-1}$	0.367879441..	0.567143291..	0.652918640..	0.704709490..		1
e_n^n	1	1.763222834..	2.345750756..	2.857390779..		∞
$e_{-n}^n = e_n^{-n}$	1	0.567143291..	0.426302751..	0.349969632..		0
$\ln_0 e_n$	1	0.567143291..	0.426302751..	0.349969632..		0
$\ln_0 e_{-n}$	-1	-0.567143291..	-0.426302751..	-0.349969632..		0
$\ln_n e_n$	1	1	1	1		1
$\ln_n e_{-n}$	-1	-1	-1	-1		-1
$\ln_0 e_n^n = W(n)$	0	0.567143291..	0.852605502..	1.049908893..		∞
$\ln_0 e_{-n}^n = W(-n)$	0	-0.567143291..	-0.852605502..	-1.049908893..		-∞
$\ln_n e_n^n$	0	1	2	3		∞
$\ln_n e_{-n}^n$	0	-1	-2	-3		-∞

Table 4

For the range $e_0 < x \text{ and } y < +\infty$, the slopes of the two derivatives diverge as x increases, and there are no real subscript functions of e_n. Note that for the range n < 0, however, according to the derivative of the natural log with respect to a change in x, the slope has imaginary sense, which generally indicates a rotation of some manner. The table above shows the results of this function for the first three integers, and an assumption of results carried to infinity. The table below shows related results with the introduction of imaginary sense to the various function.

n	$f(n)$	$n \ln_0 e_n$	$in \ln_0 e_n$	$n \ln_0 ie_n$	$in \ln_0 ie_n$
1		0.5671…	i0.5671…	0.5671…+$i\pi/2$(=+1 $i\pi/2$)	- $\pi/2$ + i0.5671…
2		0.8526…	i0.8526…	0.8526…+i π(=+2 $i\pi/2$)	- π + i0.8526…
3		1.0499…	i1.0499…	1.0499…+$i3\pi/2$(=+3 $i\pi/2$)	- $3\pi/2$ + i1.0499…
4		1.2021…	i1.2021…	1.2021…+i 2π(=+4 $i\pi/2$)	-2π + i1.2021…
5		1.3067…	i1.3067…	1.3067…+$i5\pi/2$(=+5 $i\pi/2$)	- $5\pi/2$ + i1.3067…
6		1.4324…	i1.4324…	1.4324…+i 3π(=+6 $i\pi/2$)	-3π + i1.4324…

Table 5

In the final table, it is clear that the integers, n, are the count of the rotations of ½ π and of the powers and hence the number of orders of i, both indications of a degree of orthogonal structure as in Euler, Section 0a. We are interested here specifically in the factor e_2. As a review of the above Figures hopefully makes clear, the value in the subscript exponential bases is in determining a coefficient of proportionality between two related differentials, one of which is a function of the nth-root of the logarithm to the others linear function.

The function in (1.71) finds form in the following equation, where the negative sense in the subscript has the same meaning it does in the superscript exponent, that is it represents inversion.

$$\ln_0 e_n = e_n^{-n} = e_{-n}^{n} \tag{1.74}$$

Thus for n = 2, we have the following, where it is understood that e_2 is a normalizing coefficient for any independent variable dimensional property, x,

$$\ln_0 e_2 \left(e_2\right)^2 = 1, \ \ln_0 e_{-2} \left(e_{-2}\right)^{-2} = -1 \tag{1.75}$$

and with a unit of property, x_0, and some mathematical operation and rearrangement we can state

$$\left(\ln_0 e_2\right)^{\frac{1}{2}} x_0 = e_2^{-1} x_0 = \frac{d \ln_2 e_2 x_0}{d e_2 x_0} = \frac{1}{1.53158...} \tag{1.76}$$

$$\left(\ln_0 e_{-2}\right)^{\frac{1}{2}} x_0 = i e_{-2} x_0 = \frac{-d e_{-2} x_0}{i d \ln_2 e_{-2} x_0} = \frac{-0.65291...}{i 1.0} = \frac{-1}{i 1.53158...} \tag{1.77}$$

In the above development of the neutron scale for quantum gravity (1.33) and at (1.66) we have an expression of the change in the linear scale of r_0 as the square root of the change in the natural log of the expansion stress scale,T_0. We have modeled quantum mass as a linear function of space, r_0, by the reduced angular wavelength or time by the frequency, ω_0. As developed above in connection with the Hubble rate, using the inertial constant and/or the speed of light we have

$$m_0 = f\left(H_0\right) = f\left(dr_0, dt_0\right) \tag{1.78}$$

Stress is modeled as a function of the square of both of these,

$$f_0 = f\left(dr_0^2, dt_0^2\right) = \hbar\omega_0{}^2 / r_0{}^2 = \hbar\theta / t_0{}^2 r_0{}^2 = \tfrac{\hbar}{c}\theta / t_0{}^2 r_0{}^2 \tag{1.79}$$

but as time and distance are dual values as defined by an invariant c, we can treat stress as a functional, where time is a measure, therefore a function of the change in r_0, (we could also use displacement as a measure, therefore a function of a change in t_0), as

$$f_0 = f\left(dr_0^2, c_0^2\left(dr_0^2\right)\right). \tag{1.80}$$

Thus, a change in stress with expansion leads to an increase in r_0, where a preliminary decrease in mass of the fundamental oscillation, the neutron, is equal to the mass of the emitted electron as developed above or

$$m_e = \Delta m_n = m_n\left(\frac{\delta r_0^2}{1+\delta r_0^2}\right) \tag{1.81}$$

The change in stress/energy density of the oscillation is

$$dE_1 = df_0 = f'\left(\mathrm{A}_0{}^{-2}\right) = -\frac{\hbar\omega_0^2}{\mathrm{A}_0^2} d\mathrm{A}_0 = -\frac{\tau_0}{\mathrm{A}_0^2} d\mathrm{A}_0 = -\frac{f_0}{\mathrm{A}_0} d\mathrm{A}_0 \tag{1.82}$$

where it is clear that a change in the log of the stress is inversely equal to a change in the log of the cross-section,

$$d\ln f_0 = \frac{df_0}{f_0} = -\frac{d\mathrm{A}_0}{\mathrm{A}_0} = -d\ln \mathrm{A}_0 \tag{1.83}$$

Obviously, since both stress and cross-section are unit values

$$\ln f_0 = \ln \mathrm{A}_0 \tag{1.84}$$

Thus for an exponential expansion of the cosmos, in accordance with (1.83) and using(1.77), we substitute f_0 for A_0 which is the square of $r_0 = x_0$ and get the following where *ii* indicates the symmetric and *ij* the anti symmetric or transverse components of the electron wave form

$$\left(\ln f_0\right)^{\frac{1}{2}} = ie_{-2} f_0 = \frac{-df_0}{id\ln f_0} = \frac{-de_{-2} f_{ii}}{id\ln_2 e_{-2} f_{ij}} = \frac{-0.65291... f_{ii}}{i1.0 f_{ij}} = \frac{-1.0 f_{ii}}{i1.53158... f_{ij}} \tag{1.85}$$

The imaginary sense assigned to the natural log differential is an indication of transverse motion and other energy associated with the change in stress, resulting in the change to that of the reduced Compton wavelength of the proton.

The Hamiltonian or total energy of the system resulting from beta decay is therefore the energy of the neutron, which is equal to the energy of the proton plus the energy differential of the electron wave components due to the change in stress of expansion.

$$E_0 - E_p - \left[E_e\left(df_0\right) + E_0\left(id\ln f_0\right)\right] = 0 \tag{1.86}$$

In terms of mass, as in (1.70) this is accurate to a factor of 2.16 x 10^{-7}.

$$m_n - m_p - \left[1.0 m_e - 1.53158... m_e\right] = 0 \tag{1.87}$$